

RESEARCH PROBLEMS  
FROM THE 18TH WORKSHOP ‘3IN1’ 2009

edited by Mariusz Meszka

**Abstract.** A collection of open problems that were posed at the 18th Workshop ‘3in1’, held on November 26-28, 2009 in Krakow, Poland. The problems are presented by Zdenek Ryjacek in “Does the Thomassen’s conjecture imply  $N=NP$ ?” and “Dominating cycles and hamiltonian prisms”, and by Carol T. Zamfirescu in “Two problems on bihomogeneously traceable digraphs”.

**Keywords:** Hamilton-connected graph, hamiltonian graph, dominating cycle, bihomogeneously traceable graph.

**Mathematics Subject Classification:** 05C45, 68Q25, 05C38, 05C20.

1. DOES THE THOMASSEN’S CONJECTURE IMPLY  $N=NP$ ?

Zdeněk Ryjáček  
ryjacek@kma.zcu.cz  
University of West Bohemia  
Department of Mathematics  
Czech Republic

By a *graph* we mean a simple loopless finite undirected graph  $G = (V(G), E(G))$ . A graph  $G$  is *Hamilton-connected* if  $G$  has a hamiltonian  $(x, y)$ -path for any  $x, y \in V(G)$ , and, for an integer  $k \geq 1$ ,  $G$  is  *$k$ -Hamilton-connected* if  $G - X$  is Hamilton-connected for any  $X \subset V(G)$  with  $|X| = k$ . Denote  $E^+(G) = \{xy \mid x, y \in V(G)\}$ , and for  $X \subset E^+(G)$  set  $G + X = (V(G), E(G) \cup X)$  (i.e.,  $X$  is a set of “new” edges that are “added” to  $G$ ; if  $e_1 = \{x, y\} \in E(G)$  and  $e_2 = \{x, y\} \in X$ , we consider  $e_1$  and  $e_2$  as parallel edges of  $G + X$ ). A graph  $G$  is said to be  *$k$ -edge-Hamilton-connected* if, for any  $X \subset E^+(G)$  such that  $|X| = k$  and the edges of  $X$  determine a path system, the graph  $G + X$  has a hamiltonian cycle containing all edges in  $X$ . The following facts are easy to observe.

1. A graph  $G$  is 1-edge-Hamilton-connected if and only if  $G$  is Hamilton-connected.

2. A graph  $G$  is 2-edge-Hamilton-connected if and only if:
- (i)  $G$  is 1-Hamilton-connected (i.e.,  $G - x$  is Hamilton-connected for any vertex  $x \in V(G)$ ), and
  - (ii) for any four distinct vertices  $x_1, x_2, x_3, x_4 \in V(G)$ ,  $G$  has a path factor consisting of 2 paths  $P_1, P_2$  such that both  $P_1$  and  $P_2$  have one endvertex in  $\{x_1, x_2\}$  and one endvertex in  $\{x_3, x_4\}$ .
3. If  $G$  is 2-edge-Hamilton-connected, then  $G$  is 4-connected.

Consider the following two decision problems.

**$k$ -E-HC**

**Instance:** A graph  $G$ .

**Question:** Is  $G$   $k$ -edge-Hamilton-connected?

**$k$ -E-HCL**

**Instance:** A line graph  $G$ .

**Question:** Is  $G$   $k$ -edge-Hamilton-connected?

(i.e.,  $k$ -E-HCL is  $k$ -E-HC restricted to line graphs).

**Question 1:** Determine the complexity of 2-E-HCL.

The following facts are known:

- **HAM**  
**Instance:** A graph  $G$ .  
**Question:** Does  $G$  contain a hamiltonian cycle?  
 HAM  $\in$  NPC, even if restricted to line graphs.
- **H-PATH**  
**Instance:** A graph  $G$  and distinct vertices  $u, v \in V(G)$ .  
**Question:** Does  $G$  contain a hamiltonian  $(u, v)$ -path?  
 H-PATH  $\in$  NPC, even if restricted to line graphs [1].
- **H-CONN**  
**Instance:** A graph  $G$ .  
**Question:** Is  $G$  Hamilton-connected?  
 H-CONN  $\in$  NPC [3].
- **1-H-CONN**  
**Instance:** A graph  $G$ .  
**Question:** Is  $G$  1-Hamilton-connected?  
 1-H-CONN  $\in$  NPC [6].

Thus, a common guess would be that probably 2-E-HCL  $\in$  NPC.

**Question 2:** Why is Question 1 interesting?

The following conjecture was posed in [5].

**Conjecture [Thomassen].** Every 4-connected line graph is hamiltonian.

There are many known equivalent versions of the Thomassen's conjecture; among others, we mention the following.

**Theorem.** The following statements are equivalent:

- (i) Every 4-connected line graph is hamiltonian.

- (ii) Every 4-connected line graph is 2-edge-Hamilton-connected [4].
- (iii) Every snark has a dominating cycle [2].

Thus, if the Thomassen’s conjecture is true, then a line graph  $G$  is 2-edge-Hamilton-connected if and only if  $G$  is 4-connected, implying that 2-E-HCL is polynomial. Consequently, proving the “common guess”  $2\text{-E-HCL} \in \text{NPC}$  would mean

- disproving the Thomassen’s conjecture,
- proving the existence of a snark with no dominating cycle,

unless  $P=NP$ .

## REFERENCES

- [1] A.A. Bertossi, *The edge hamiltonian path problem is NP-complete*, Inform. Process. Lett. **13** (1981), 157–159.
- [2] H.J. Broersma, G. Fijavž, T. Kaiser, R. Kužel, Z. Ryjáček, P. Vrána: *Contractible subgraphs, Thomassen’s conjecture and the dominating cycle conjecture for snarks*, Discrete Mathematics **308** (2008), 6064–6077.
- [3] Alice M. Dean, *The computational complexity of deciding Hamiltonian-connectedness*, Proceedings of the Twenty-fourth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1993). Congr. Numer. **93** (1993), 209–214.
- [4] R. Kužel, Z. Ryjáček, P. Vrána, *On some conjectures on hamiltonian properties of 4-connected line graphs*, Preprint 2009, to appear.
- [5] C. Thomassen, *Reflections on graph theory*, J. Graph Theory **10** (1986), 309–324.
- [6] P. Vrána, personal communication.

## 2. DOMINATING CYCLES AND HAMILTONIAN PRISMS

Zdeněk Ryjáček  
 ryjacek@kma.zcu.cz  
 University of West Bohemia  
 Department of Mathematics  
 Czech Republic

The *prism over a graph  $G$* , denoted  $G \square K_2$ , is the Cartesian product of  $G$  and  $K_2$ . It consists of two disjoint copies of  $G$  and a perfect matching connecting a vertex in one copy of  $G$  to its “clone” in the other copy.

A graph  $G$  is *hamiltonian* if it has a hamiltonian cycle and *traceable* if it has a hamiltonian path. Define a *k-walk* in a graph to be a spanning closed walk in which every vertex is visited at most  $k$  times

The following implications are easy to verify:

$G$  is hamiltonian  $\Rightarrow G$  is traceable  $\Rightarrow G \square K_2$  is hamiltonian  $\Rightarrow G$  has a 2-walk.

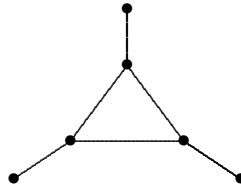
Thus the question whether  $G$  has a hamiltonian prism (i.e whether  $G \square K_2$  is hamiltonian) is “sandwiched” between hamiltonicity and having a 2-walk. Specifically, the property of having a hamiltonian prism can be considered as a “relaxation” of hamiltonicity. More information about prism-hamiltonicity of a graph can be found e.g. in [1] and [2].

A *dominating cycle* in a graph  $G$  is a cycle  $C$  such that every edge of  $G$  has at least one vertex on  $C$ , i.e. such that the graph  $G - C$  is edgeless. Clearly, a hamiltonian cycle is dominating, and hence the property of having a dominating cycle can be considered as another relaxation of hamiltonicity.

There is a natural question whether there is any relation between these two properties.

**Example 1.** Let  $H$  be any 2-connected cubic nonhamiltonian graph, and let  $G$  be obtained from  $H$  by replacing every vertex of  $H$  with a triangle (such a  $G$  is sometimes called the *inflation* of  $H$ ). Then  $G$  is a 2-connected line graph and these are known [2] to be prism-hamiltonian. On the other hand, since  $H$  is nonhamiltonian, any cycle in  $G$  has to miss at least one “new” triangle and hence  $G$  has no dominating cycle. Thus, there are “many” graphs showing that hamiltonian prism does not imply having a dominating cycle.

**Example 2.** The graph in the figure below shows that also the existence of a dominating cycle does not imply having hamiltonian prism.



However, all such known examples are of low toughness (recall that  $G$  is 1-tough if, for any  $S \subset V(G)$ , the graph  $G - S$  has at most  $|S|$  components). This motivates the following question.

**Conjecture.** *Let  $G$  be a 1-tough graph having a dominating cycle. Then  $G$  has hamiltonian prism.*

**Comments.** Suppose that  $G$  has a dominating cycle  $C$  of even length. Set  $M = V(G) \setminus V(C)$  and  $N = \{x \in V(C) \mid x \text{ has a neighbor in } M\}$ . Then the graph induced by  $M \cup N$  has a matching containing all vertices from  $M$  (this follows by the toughness assumption and by the Hall’s theorem). Using this matching, it is easy to construct a hamiltonian cycle in  $G \square K_2$ .

The difficult case is when all dominating cycles in  $G$  are of odd length.

## REFERENCES

- [1] R. Čada, T. Kaiser, M. Rosenfeld, Z. Ryjáček, *Hamiltonian decompositions of prisms over cubic graphs*, Discrete Mathematics **286** (2004), 45–56.
- [2] T. Kaiser, D. Král, M. Rosenfeld, Z. Ryjáček, H.-J. Voss, *Hamiltonian cycles in prisms over graphs*, Journal of Graph Theory **56** (2007), 249–269.

## 3. TWO PROBLEMS ON BIHOMOGENEOUSLY TRACEABLE DIGRAPHS

Carol T. Zamfirescu  
 czamfirescu@gmail.com  
 Technische Universität Dortmund  
 Vogelpothsweg 87, 44221 Dortmund  
 Germany

We concern ourselves here exclusively with simple finite oriented graphs (i.e. digraphs with no multiple edges, a finite number of vertices, and without cycles of length 2), calling these simply *graphs*. A graph is called *homogeneously traceable*, if for every vertex  $v$  there exists a hamiltonian path starting at  $v$ . If, additionally, the graph has the property that in every vertex a hamiltonian path ends, we call it *bihomogeneously traceable*. In this setting, and in a graph on  $n$  vertices, *arc-minimality* (or 2-regularity) means that the graph has precisely  $2n$  edges (i.e. every vertex has in-degree 2 and out-degree 2). We remark that bihomogeneous traceability does not imply hamiltonicity, for instance hypohamiltonian graphs are non-hamiltonian and bihomogeneously traceable.

Z. Skupień [3] presented in 1981 an infinite family of arc-minimal non-hamiltonian bihomogeneously traceable graphs, featuring graphs of all orders greater or equal to 7. Another such infinite family of graphs (but not arc-minimal) was provided independently by S. Hahn and T. Zamfirescu [1] in the same year.

In 1983, L. E. Penn and D. Witte [2] proved that the cartesian product of two oriented cycles of length  $a$  and  $b$  is hypohamiltonian (whence, non-hamiltonian and bihomogeneously traceable) if and only if there exist relatively prime numbers  $m, n \in \mathbb{N}$  such that  $am + nb = ab - 1$ . We note that these graphs are also arc-minimal.

In their 1981 paper, Hahn and Zamfirescu presented two planar non-hamiltonian bihomogeneously traceable graphs, one of which is arc-minimal, and asked the natural question whether infinitely many such graphs do exist. Very recently it was proven that this is indeed the case, see [4].

The following problems, however, are still open.

**Problem 1.** Is there an infinite family of planar arc-minimal non-hamiltonian bihomogeneously traceable oriented graphs?

**Problem 2.** Are there such graphs on all orders greater than some integer? Even if one removes the condition of arc-minimality, this problem is still open.

## REFERENCES

- [1] S. Hahn, T. Zamfirescu, *Bihomogeneously traceable oriented graphs*, Rend. Sem. Mat., Torino **39** (1981) 2, 137–145.
- [2] L.E. Penn, D. Witte, *When the Cartesian product of two directed cycles is hypo-Hamiltonian*, J. Graph Theory **7** (1983), 441–443.
- [3] Z. Skupień, *On homogeneously traceable nonhamiltonian digraphs and oriented graphs*, Proc. Fourth Int. Conf. on the Theory and Appl. of Graphs, Kalamazoo (MI) 1980, Wiley (1981), 517–527.
- [4] C.T. Zamfirescu, *An infinite family of planar non-hamiltonian bihomogeneously traceable oriented graphs*, to appear in Graphs and Combinatorics.

Mariusz Meszka  
meszka@agh.edu.pl

AGH University of Science and Technology  
Faculty of Applied Mathematics  
al. Mickiewicza 30, 30-059 Cracow, Poland

*Received: November 28, 2009.*

