A Planar Hypohamiltonian Graph with 48 Vertices

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Abstract: We present a planar hypohamiltonian graph on 48 vertices, and derive some consequences. J Graph Theory 55: 338–342, 2007

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1. INTRODUCTION

A graph is called *hypohamiltonian* if it is not hamiltonian but, when omitting an arbitrary vertex, it becomes hamiltonian. The smallest hypohamiltonian graph is the famous Petersen graph (from 1898) with 10 vertices. For quite a while the existence of planar hypohamiltonian graphs was ignored, until Thomassen [3] discovered in 1976 an infinite family of such graphs, the smallest among them weighing in at 105 vertices. Hatzel [1] improved this lower bound to 57 vertices in 1979. In this article, we present a planar hypohamiltonian graph with 48 vertices only.

Gallai formulated in 1966 the question whether, contrary to the case of cycles, in any finite connected graph one can find a vertex which lies on all paths of maximal length.

This question was answered (in the negative) by Walther [6], and later refined in [7]. A small planar cubic hypohamiltonian graph found by Thomassen [4] and our graph presented here will be used to improve several bounds asked for in [7]. Also, we provide a new, rather small planar hypotraceable graph.

2. A PLANAR HYPOHAMILTONIAN GRAPH

We present here our planar hypohamiltonian graph. It has 48 vertices.

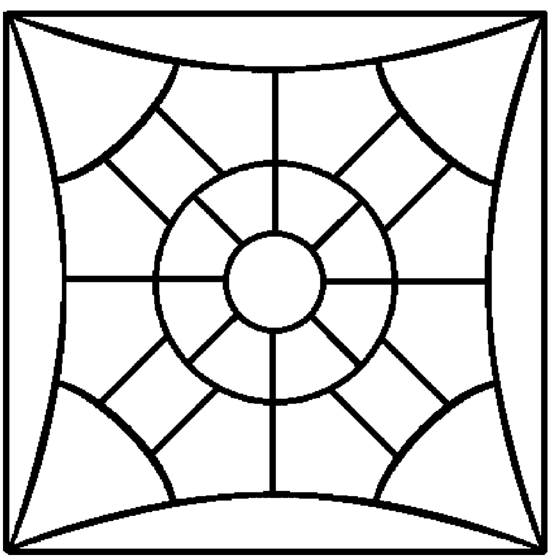
Theorem. *The graph below is planar and hypohamiltonian.*

Proof. Let G be the depicted graph. For the reader's convenience, let us recall Grinberg's hamiltonicity criterium for planar graphs. It says that, given a planar graph with a hamiltonian cycle C and with f_i i-gons inside and f'_j j-gons outside of C, we have

$$\sum_i (i-2)(f_i - f'_i) = 0.$$

Thus, if in a graph embedded in the plane every face but one is an i-gon with $i = 2$ modulo 3, then the graph is not hamiltonian. Since the planar graph G has one quadrilateral and otherwise only pentagons and octagons as faces, it is not hamiltonian.

Since G has an automorphism group of size 8 with 9 orbits, Figures 1 to 9 suffice to show that every vertex is missed by some cycle of length 47. ■



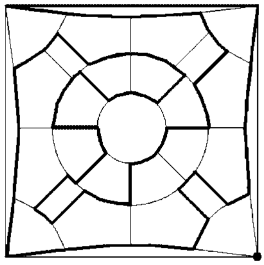


FIGURE 1.

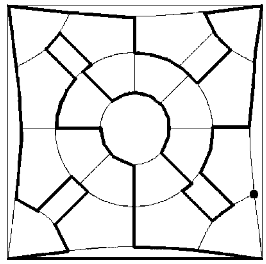


FIGURE 2.

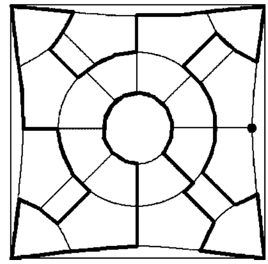


FIGURE 3.

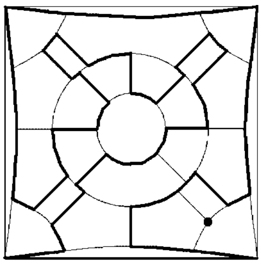


FIGURE 4.

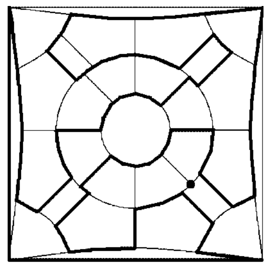


FIGURE 5.

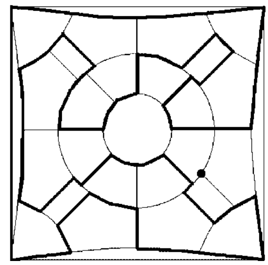


FIGURE 6.

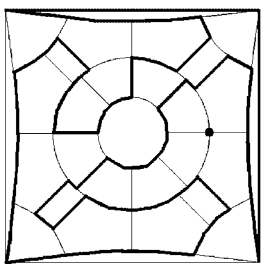


FIGURE 7.

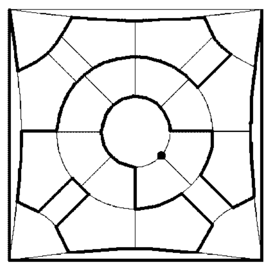


FIGURE 8.

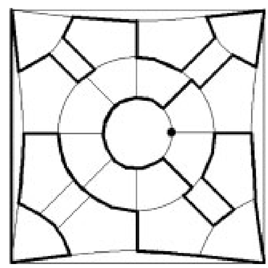


FIGURE 9.

3. CONSEQUENCES

If we replace in the definition of hypohamiltonicity "hamiltonian" by "traceable", we obtain the notion of *hypotraceability*.

C. Thomassen described in [4] a method of constructing a hypotraceable graph starting with four (possibly isomorphic) hypohamiltonian ones. This method also preserves planarity if the four hypohamiltonian graphs are all planar. So, we can use it and derive a relatively small planar hypotraceable graph.

Corollary 1. *There exists a planar hypotraceable graph on 186 vertices.*

Proof. We use the graph G from the proof of the Theorem. Choose a vertex $v \in V(G)$ of degree 3 and the neighbors x, y, z of v in G. Let G_1, G_2, G_3, G_4 be 4 isomorphic copies of G, and denote by v_i, x_i, y_i, z_i the points in G_i corresponding to v, x, y, z $(i = 1, 2, 3, 4)$.

Consider $G_i - v_i$, the graph G_i with the vertex v_i removed ($i = 1, 2, 3, 4$), identify x_1 with x_2 and x_3 with x_4, and join y_1 with y_3, z_1 with z_3, y_2 with y_4 and z_2 with z_4. The resulting graph is by a theorem of Thomassen in [4] hypotraceable.

It has 186 vertices.

Let $\overline{C}_k^j$ ($\overline{P}_k^j$) be the smallest number of vertices of a planar k-connected graph, in which any j vertices are omitted by some longest cycle (path).

Tables I and II on page 79 in Voss' book [5], and also the survey [9], show the following inequalities: $\overline{C}_3^1 \leq 57$; $\overline{P}_3^1 \leq 224$; $\overline{C}_3^2 \leq 6758$; $\overline{P}_3^2 \leq 26378$. ■

These inequalities can be improved using our Theorem.

Corollary 2. *We have:*

1. $\overline{C}_3^1 \leq 48$
2. $\overline{P}_3^1 \leq 188$
3. $\overline{C}_3^2 \leq 4277$
4. $\overline{P}_3^2 \leq 16926$.

Proof.

1. This follows immediately from the Theorem.
2. Replace in the complete graph K_4 every vertex with a graph which is created by removing a vertex of degree 3 from G. We obtain a planar graph G' with the property that every vertex is omitted by a path of maximal length. G' is 3-connected and has 188 vertices; thus we obtain: $\overline{P}_3^1 \leq 188$.
3. To prove $\overline{C}_3^2 \leq 4277$ we "open" again our graph G at a vertex of degree 3 and introduce it into each vertex of Thomassen's graph H in [4], p. 38, following the same old idea from [8]. Since every pair of edges in H is missed by a longest cycle (see [2]), the resulting graph has the property that any two vertices are missed by some longest cycle. This property is not lost if we then contract all edges which originally belonged to H.
4. For the last inequality we proceed similarly, but take first the graph H' obtained by "opening" H at a vertex and introducing it into each vertex of K_4, and then introduce at each vertex of H' a copy of G "opened" at some vertex of degree 3. Eventually we contract all edges which originally belonged to H'. ■

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